Analysis of Voter Behavior in Liquid Democracy

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1 Introduction

Liquid democracy is a recently developed approach to voting which allows an individual to either directly vote or delegate her vote to anyone in their social network. It can be thought of as a mixture of direct democracy (where each individual has a choice to directly vote) and representative democracy (where a set of delegates representing a society votes on issues). Liquid democracy was introduced to try to increase voter turnout and to try to lessen the problems of both direct and representative democracies.

One underlying factor for low voter turnout of direct democracy relates to costs of voting. When a group of voters is large, the probability that an individual vote changes the result of an election is essentially zero. Under such circumstances, it stands to reason that individuals with relatively large costs of voting might not vote at all. This is supported by Figure 1, which shows reasons why people did not vote between 2000-2016.
When do relatively large costs of voting become problematic for the goals of direct democracy? If costs were randomly distributed among voters independent of demographics, then (assuming a large enough population of voters) voters with low costs would represent the interests of the population as a whole. As such, direct democracy would be a satisfactory enough voting system to use. However, it may not be reasonable to assume costs are distributed randomly among voters independent of their demographics. As seen in Figure 2, we see voter turnout in the U.S. presidential election is the highest among White voters compared to Black, Asian, and Hispanic voters. However, we also see that White voters tend to lean more Republican whereas Black, Asian, and Hispanic voters tend to lean more Democratic. Therefore, perhaps one can assume that the cost of voting among nonwhite demographics is higher. If this assumption is true, then a large proportion of Democratic votes could be lost resulting in a voting turnout that is not representative of the society as a whole.
Representative democracy, although practical, also has implementation issues. Specifically, representative democracy is susceptible to corruption which, according to Ferreira (2017), is defined as “...the observable part of the big deal... set up by the economic conglomerate players in the election market game”. Therefore, representatives voting on behalf of citizens might be swayed into voting against what they truly prefer, which is not beneficial to society as a whole.

Ideally, liquid democracy improves upon direct and representative democracy. For individuals who cannot vote directly, it now gives them the opportunity to delegate and exercise their voting power. Furthermore, by allowing individuals to choose a delegate in their social network, it will hopefully decrease the effect of corruption among delegates given the pool of delegates will become very large.

However, applying liquid democracy in national politics involves practical and philosophical challenges that may prove to be insurmountable (Cheeseman, Lynch, and Willis (2018)). Nonetheless, liquid democracy can provide a much fairer decision procedure when a large group with an apathetic majority has to agree on a decision regarding highly complex issues. For example, Boldi et al. (2011) suggests that liquid voting will arrive to a more sensible outcome when deciding on new terms of user
agreement of a website. Furthermore, shareholders’ voting process can also be improved by the liquid democracy mechanism.

In our paper, we analyze how voters should act in terms of their voting costs and perceived influence. We hypothesize that a voter will vote directly if she has a very low cost or if she has a high perceived influence relative to her neighbors. On the other hand, a voter will delegate if she has a high cost and has a neighbor who has a low cost or has a high probability of knowing someone with a low cost.

Our paper proceeds as follows. In section 2, we discuss other relevant literature. In section 3, we discuss in more detail why individuals have a very low perceived influence in a direct democracy setting. In section 4, we formalize liquid democracy in order to create models which are discussed in section 5 and section 6. In section 7, we provide simulations for our proposed model. Finally, in section 8 we provide concluding remarks.

2 Literature Review

The obvious limitations of standard voting schemes have inspired several researchers (Miller (1969), Tullock (1992)) to look into using liquid democracy as an alternative. The emergence of information technology has heightened the anticipation that liquid democracy can begin to make real contributions to making collective decision processes more fair. This is because liquid democracy is a more difficult computational problem compared to other voting schemes. In addition, as Brill (2018) notes, the remarkable research advancements in the intersection of economics and computer science have provided researchers with concepts and techniques to aid the design process of liquid democracy.

There are only a few theoretical publications of liquid democracy which all have been published in recent years. The paper that resembles our work the most is Kahng, Mackenzie, and Procaccia (2018), in which the authors create a model based on voters with varying “competence levels,” or probabilities of voting their true preferences. They find that, without central coordination, only delegating to those with strictly higher competence levels does not lead to a favorable outcome mainly due to concentration of voting weights on a few voters. In our work, we assume that if a voter endures a cost of research that she will be correct in how she votes. Then the individual will make delegation decisions based on her neighbors’ similarities to her and also based on her likelihood of eventually passing her
vote to a candidate.

3 Game theory of direct democracy

Low voter turnout is an undesirable yet prevalent phenomenon in democratic societies. One important explanation for the reluctance of individuals to vote is diffusion of responsibility. The limited influence of an individual causes masses of voters to feel that their vote cannot affect the outcome, resulting in a phenomenon similar to “the bystander effect.”

To understand this issue in a game theoretical setting, suppose there are \( n + 1 \) total eligible voters (assume \( n \) is even for simplicity). Isolate voter 1, \( V_1 \) and assume she prefers candidate 1, \( C_1 \). Furthermore, suppose that a proportion \( p \) of the population prefers \( C_1 \) over all other candidates. For a given voter \( V_i \), if her preferred candidate wins, then she gets a utility of 1. Otherwise, she gets a utility of \( -1 \). Lastly, assume that the number of votes candidate \( C_1 \) gets is \( X_1 \sim Bin(n, p) \) (where \( V_i \) doesn’t vote yet).

Every voter has to choose between voting and abstaining. If a voter has a positive cost of voting, then she may only want to vote if she thinks her vote will “matter.” This means that she perceives that her vote will play a deciding role in the election. A table of how an individual’s vote affects the election can be shown below (using \( V_1 \) as an example):

<table>
<thead>
<tr>
<th>( V_1 ) vote</th>
<th>less than ( \frac{n}{2} ) vote ( C_1 )</th>
<th>exactly ( \frac{n}{2} ) vote ( C_1 )</th>
<th>more than ( \frac{n}{2} ) vote ( C_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>abstain</td>
<td>(-1)</td>
<td>(+1)</td>
<td>(+1)</td>
</tr>
<tr>
<td>vote</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(+1)</td>
</tr>
</tbody>
</table>

Table 1: Payoff table for \( V_1 \)

Now, returning to our problem stated two paragraphs above, what can we say about the probability \( X_1 = \frac{n}{2} \)? Formally, \( Pr(X_1 = \frac{n}{2}) \rightarrow 0 \) as \( n \rightarrow \infty \) (proof in Appendix A). Therefore, if \( V_1 \) has a positive cost of voting and she believes her vote will not change the election, then she most likely will not want to incur this voting cost. Thus, she will abstain from voting in direct democracy.
Thus, if we include various costs involved in voting such as doing research and transportation to polling locations, voting may not be a dominant strategy anymore. Therefore, under a game theoretic sense, we conclude that individuals’ decision to abstain from voting and sparing themselves of the voting costs is hardly irrational. We use liquid democracy to free those that would endure high costs of directly voting by delegating their vote to an individual in their social network.

4 Formalization of the Problem

Our model is comprised of a (social network) directed graph $G((V, O), \bar{E})$, where $O$ and $V$ represent candidates and voters respectively. Assuming that we have $N$ voters and $K$ candidates in this single-stage game, $C_i$ is defined as the net cost incurred by direct voting of $V_i$ to $O_k$. We assume that this cost is unique for every voter $V_i$ regardless of which candidate she votes for. According to Downs (1957) and Riker and Ordeshook (1968), the “net cost” of voting could be negative as it can be formalized as:

$$C_i = R_i + T_i - H_i$$ (1)

where $R_i$ is the research cost (i.e., the cost of researching the candidates and making an informed decision), $T_i$ is the transportation cost (i.e., the cost of physically going to the polling booth and voting), and $H_i$ is citizen duty (i.e., the goodwill/psychological feeling and perceived civic benefit of voting). When voters have a higher benefit from voting directly than the costs associated with voting directly, their cost is negative. So, they would vote directly in both direct and liquid democracy.

A directed edge from voter $V_i$ to voter $V_j$, i.e. $E(V_i, V_j)$ exists if $V_i$ knows $V_j$. $V_i$ is allowed to delegate to $V_j$ only if this edge exists. Note that since the edges are directed, $V_i$ might be allowed to delegate to $V_j$ but $V_j$ might not be allowed to delegate to $V_i$. This is because individuals can delegate to well-known personalities but these personalities may not know the people delegated to them. We can now define a set of delegates $D(V_i)$ for each voter $V_i$ that she can delegate her vote to such that $V_j \in D(V_i)$ if the edge $E(V_i, V_j)$ exists for $j \neq i$.

For each voter $i \in V$, we have $u_i(s_i, s_{-i})$ as the utility of $V_i$ if she plays the strategy $s_i$ and the rest

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1 That is, $|O| = K, |V| = N$.

2 Even if every voter has implicit preferred candidates, a voter can not vote her preferred candidate directly without doing research.
of the voters play $s_i$ where $s_i \in \{O_1, O_2, \ldots, O_K, D(V_i), Abstain\}$, i.e., $V_i$ could vote directly for some candidate $O_k$, delegate to another voter, or abstain from voting. We define utility as the gain from the voting result minus the cost of participation:

$$u_i(s_i, s_{-i}) \triangleq g_i(O_w) - c_i(s_i)$$

where $O_w$ is the final candidate determined by social choice procedure (e.g. winner of the election):

$$O_w = f(s_i, s_{-i})$$

and

$$g_i : O_k \rightarrow \mathbb{R}, \forall k\quad (4)$$

$$c_i : s_i \rightarrow \mathbb{R} \forall i\quad (5)$$

where $g_i$ is the gain function for $V_i$ and $c_i$ is the cost function for $V_i$. Assume that $V_i$ has information of all $g_j$ and $C_j$ for $V_j \in D(V_i)$ (we say “$V_i$ knows $V_j$ well”).

5 Baseline Model

We first simplify our model of liquid democracy to see if we can find equilibria. From this simplified framework, we will present our baseline model. For each voter $V_i$ and strategy $s_i$, we assign costs as follows:

$$c_i(s_i) = \begin{cases} 
C_i \sim F(\cdot) \text{ and } C_i \in [a, b] & \text{if } s_i \in \{O_1, O_2, \ldots, O_K\} \\
0, \text{ otherwise}
\end{cases}$$

Note that this implies that for each voter, only direct voting can have a nonzero cost. So, we assume for now that delegating and abstaining bears zero cost. Also, note that $a$ can be less than 0 meaning that the voter gains utility from directly voting (see Equation 1). Lastly, $F(\cdot)$ is some probability distribution.
We assign gain for each voter as

\[ g_i(O_k) = \begin{cases} 
1, & \text{if } O_k = O_{w_i}, w_i \in \{1, 2, \cdots, K\} \\
0, & \text{otherwise}
\end{cases} \]

This assumption indicates that a voter gains from the voting result if and only if her preferred candidate \( O_{w_i} \) wins. We assume that every voter has an implicit preferred candidate, but a voter does not know this preferred candidate without doing research. Since voters prefer only one candidate, we can define a “base” \( B_k \) of candidate \( O_k \) to the set of voters who prefer candidate \( O_k \). From how we defined our gain function, it is obvious these sets are disjoint.

We further assume that voters know the out-degrees \( \text{deg}^+(V_j) \) for all \( V_j \in D(V_i) \).

Lastly, the winner of this voting system is determined by the function \( f(s_i, s_{-i}) \). Our voting rule is an extended form of plurality. This means that if voter \( V_i \) delegates their vote to \( V_j \), \( V_j \)'s voting weight will increase by 1.

In our model, we want \( V_i \) to maximize \( \Delta E[u_i] = E[u_i(s_i)] - E[u_i(Abstain)] \) where \( s_i \in \{\text{Direct Vote, Delegate}\} \). Intuitively, we want her expected utility of either directly voting or delegating to be as large as possible compared to her expected utility to abstain. We can think of \( \Delta E[u_i] = 1 \times \mathcal{P}(\text{vote matters}) + 0 \times \mathcal{P}(\text{vote doesn’t matter}) - C_i \times 1 \) [vote directly]. By “probability of vote matters” we mean that it is the probability that sending one’s vote through direct voting or delegating will impact the election. We formalize it as follows:

\[
\Delta E[u_i] = E[u_i(s_i)] - E[u_i(Abstain)] \\
= E[g_i(s_i)] - E[g_i(Abstain)] - C_i \times 1 \text{ [vote directly]} \\
= 1 \times \mathcal{P}(\text{vote matters}) - C_i \times 1 \text{ [vote directly]} \\
= 1 \times \mathcal{P}(\text{vote counted}) \times \mathcal{P}(\text{vote matters} | \text{vote counted}) \\
- C_i \times 1 \text{ [vote directly]}
\]

(7)
Thus, we have

\[
\Delta E[u_i] = \begin{cases} 
\mathbb{P}(\text{vote matters}|\text{vote counted}) - C_i & \text{if vote directly} \\
\mathbb{P}(\text{vote counted}) \times \mathbb{P}(\text{vote matters}|\text{vote counted}) & \text{if delegate} \\
0 & \text{if abstain}
\end{cases}
\]  

Even with these assumptions, voters can not calculate the probability of their votes “mattering” explicitly and, thus, we cannot find a Nash equilibrium. If the probability of a vote mattering is small compared to \(|C_i|\), then the randomly assigned costs \(C_i\) tend to dominate voters’ decision-making on how to act. As a result, the following is our initial proposed algorithm:

1. If \(V_i\) has \(C_i \leq 0\), \(V_i\) votes for \(O_{u_i}\) directly.

2. If \(V_i\) has \(C_i > 0\) and has a neighbor \(V_j\) in the same base as herself with non-positive cost, then \(V_i\) will delegate to \(V_j\).

3. Otherwise, \(V_i\) delegates her vote to the voter in the same base as herself with a highest out-degree.

If a voter has an non-positive cost (civil duty surpasses research and transportation costs), it makes sense to vote directly because it is the least costly strategy. Now, if a voter \(V_i\) has a positive cost, delegating to the neighbor \(V_j\) with non-positive cost makes sense because we know \(V_j\) will vote directly making \(\mathbb{P}(\text{vote counted}) = 1\) and, so, \(V_i\) will get maximum utility from delegating. Otherwise, \(V_i\) will give it to a neighbor \(V_j\) with highest out-degree, hoping either that \(V_j\) will be able to find a voter with non-positive cost or will pass the vote to another voter with a high out-degree. This means that voters with positive costs collaborate to look for voters with non-positive costs since they know that only voters with non-positive cost will vote directly.

6 Extended Model

In our baseline model in section 5, we assume \(\mathbb{P}(\text{vote matters})\) is likely to be negligible compared with \(C_i\). This leads to a problem that only a fixed number of voters will vote directly (the same voters in direct democracy). Every voter with a positive cost seeks to become a free-rider to avoid costs even if
she is highly influential in her network. In order to consider self-perceived influences, we need to pay more attention on the estimation of $E[g_i]$. From Equation 8, we have:

$$
\Delta E[u_i] = \begin{cases} 
\mathcal{P}(\text{vote matters}|\text{vote counted}) - C_i & \text{if vote directly} \\
\mathcal{P}(\text{vote counted}) \times \mathcal{P}(\text{vote matters}|\text{vote counted}) & \text{if delegate} \\
0 & \text{if abstain}
\end{cases}
$$

We further assume that voter $V_i$ knows her in-degree $\deg^-(V_i)$ indicating she knows her influence. Since it is impossible to calculate explicitly the probability that a voter’s vote changes the voting result, we want to approximate $\mathcal{P}(\text{vote counted})$ and $\mathcal{P}(\text{vote matters}|\text{vote counted})$. There is no doubt that $\mathcal{P}(\text{vote matters}|\text{vote counted})$ should depend on her in-degree $\deg^-(V_i)$ and $N$ (the number of voters in the whole network). This is because the higher a voter’s in-degree, the more delegations she is able to receive and, therefore, the more votes she can award her preferred candidate. The greater this number is compared to the total number of voters, the greater a voter’s $\mathcal{P}(\text{vote matters}|\text{vote counted})$. Therefore:

$$
\mathcal{P}(\text{vote matters}|\text{vote counted}) \approx \frac{\deg^-(V_i)}{N} \quad (9)
$$

The above equation is the proportion of voters who know $V_i$ over total population, which is a proxy of how influential $V_i$ is. Moreover, we suppose that voters’ perceived $\mathcal{P}(\text{vote counted})$ when delegating to $V_j$ is approximately:

$$
\mathcal{P}(\text{vote counted}) \approx 1[C_j \leq 0] + 1[C_j > 0] \times \frac{\deg^+(V_j)}{N} \quad (10)
$$

If $V_j$ has a non-positive cost, $V_i$ knows that $V_j$ will vote on her behalf certainly. If there does not exist a neighbor $V_j \in D(V_i)$ such that $C_j \leq 0$, $V_i$ delegates knowing there is a risk her vote might not reach her preferred candidate. To lower the risk, $V_i$ considers delegating to $V_j \in D(V_j)$ with highest out-degree $\deg^+(V_j)$. This is similar to our baseline model\(^1\) in section 5. Nonetheless, now $V_i$ needs

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\(^1\) Notice that theoretically $\mathcal{P}(\text{vote counted})$ also increases with $\deg^-(V_j)$ and decreases with $C_j$, which is not embodied in Equation 10. However, according to our model, $\deg^+(V_j)$ plays a much more important role than $\deg^-(V_j)$ and $C_j$ in $\mathcal{P}(\text{vote counted})$. 
to make a trade-off between voting directly and delegating based on:

$$\Delta E[u_i] = \begin{cases} 
\frac{\text{deg}^-(V_i)}{N} - C_i & \text{if vote directly} \\
(1 [C_j \leq 0] + 1 [C_j > 0] \times \frac{\text{deg}^+(V_i)}{N}) \times \frac{\text{deg}^-(V_i)}{N} & \text{if delegate} \\
0 & \text{if abstain}
\end{cases}$$

(11)

The solution of $\max \Delta E[u_i]$ here is congruent with what we have in baseline model, but has taken into account the influence among voters. Voters now can calculate Equation 11 explicitly to optimize their behaviors which is what we do in the simulations that follow.

7 Simulations

We use simulations to determine whether liquid democracy can outperform direct democracy. We give two criteria for performance: “voter representation” and “accuracy.” We define voter representation as the proportion of votes that are counted (i.e., reach a candidate) relative to the total number of voters. Note that in liquid democracy, a vote that is delegated, but never reaches a candidate, is not counted. It is generally agreed that higher voter representation is better.

Also, we observe whether the votes counted produce the ideal candidate and define this as accuracy. We measure this by simulating a specific voting mechanism multiple times and seeing the percentage of times the voting mechanism produces the ideal winner. Obviously, higher accuracy is desirable.

The simulation was run as follows. We choose $N$ voters and $K$ candidates. Each voter is assigned a preferred candidate uniformly at random. We assume that each voter prefers exactly one candidate. Then each voter is assigned a cost distributed Uniform($-1, 1$). Using these voters, the winning candidate is determined using direct democracy or liquid democracy. Following are the specifics of the voting mechanisms.

In direct democracy, each voter either votes directly or abstains. For each voter, if her cost is less than or equal to 0, she will vote directly for her preferred candidate. Otherwise, she will abstain. The candidate who receives the highest number of votes wins.
In liquid democracy, we generate a graph that follows the Barabási-Albert model. Under the Barabási-Albert model, most voters have only a few incoming and outgoing edges, but a few voters tend to accumulate a large number of edges. More details about the Barabási-Albert graph can be found at wikipedia. We use this model because it is an approximate representation of social networks.

![Figure 3: Barabási–Albert Model](image)

When a voter is initially added she will be connected to at most 2 other voters who are already in the graph (specifically, $m_0 = 2$). Once all edges between voters are created, we can run our algorithm. The voters in this graph are the same as in the direct democracy but now they have a social network connecting them to each other. Also, voters act differently in liquid democracy than in direct democracy. Namely, any voter $V_i$ with $C_i \leq 0$ will vote directly for her preferred candidate, giving this candidate her vote plus all votes delegated to her. For voter $V_i$ with positive cost, she will seek to maximize $\Delta E[u_i]$. Recall that

$$
\Delta E[u_i] = \begin{cases} 
\frac{\deg^{-}(V_i)}{N} - C_i & \text{if vote directly} \\
(1 \cdot [C_j \leq 0] + 1 \cdot [C_j > 0]) \times \frac{\deg^{+}(V_j)}{N} & \text{if delegate} \\
0 & \text{if abstain}
\end{cases}
$$

(12)

Each voter with positive cost will therefore choose the action which maximizes $\Delta E[u_i]$. The winning candidate is determined by the plurality rule.

Simulations were run in Python. Due to the computational complexity of the generation of the Barabási-Albert graph and the liquid voting algorithm, we used $N = 1,000$ with $k = 5$. We ran 10 trials. That is, voters were reinitialized 10 times and the same voters were used for direct democracy and liquid democracy in order to make comparisons between the different models. The results are as follows. Liquid democracy increased voter representation by an average of 23%. Liquid democracy
produced the ideal winner 70% of the time while direct democracy produced the ideal winner 40% of the time. Notably, for any given trial, there was no instance in which direct democracy produced the ideal winner and liquid democracy did not. These results support our hypothesis that liquid democracy improves upon direct democracy since voter representation and accuracy increases.

8 Conclusion

Liquid democracy is a relatively new line of research and still has a lot of room for further insight. Our goal was to analyze behavior of voters in liquid democracy compared to direct democracy. Through simulation, we found that our model setup suggests that liquid democracy increases voter representation and accuracy compared to direct democracy. These are desirable results and support our hypothesis that the liquid democracy model improves upon direct democracy.

There are still many extensions that can be produced from our work. First, although we did define representative democracy in our introduction, we did not compare it to liquid democracy. Furthermore, some of the simplifications we made in our model can be generalized. For example, we viewed “similarity” between two voters as binary (1 if the voters prefer the same candidate, 0 otherwise). A possible next step to our work would be to define similarity between voters on a scale from 0 to 1. This is because it is unlikely to be perfectly similar to someone and, thus, delegating bears the risk that your delegatee will actually vote against your own preferences. Another possibility for extension is considering a non-binary gain function. Lastly, in our final model, when voters were considering delegation, they would only observe their neighbors’ costs and out-degrees. Perhaps other factors could be important to observe as well such as their neighbors’ in-degrees.

Furthermore, it would be interesting to investigate whether powerful firms or politicians would bribe individuals into selling their votes to them in order to impact the election in their favor. If liquid democracy were implemented in practice, this would likely be a concern.

Lastly, note that in our simulation we used uniformly distributed costs. As stated in the introduction, one motivation of using liquid democracy is that generally costs are not randomly distributed among voters. Nonetheless, our simulation still gave us good insight on how our liquid democracy model behaves in comparison to direct democracy. In the future, it would be interesting to run this simulation again with a different values of $N$ and $m_0$. Furthermore, it would be interesting to “skew”
costs and correlate them with voters of different preferences.

It is well known that there are many problems with our current voting systems. Hopefully, more progress can be made in the research of liquid democracy that can help improve on these voting systems to ensure fairness.
A Appendix

Claim: Given $X \sim \text{Bin}(n, p)$ where $n$ is even, then $Pr(X = \frac{n}{2}) \to 0$ as $n \to \infty$.

Proof: We know $Pr(X = \frac{n}{2}) = \binom{n}{\frac{n}{2}} p^{\frac{n}{2}} (1 - p)^{\frac{n}{2}}$. We need to show $Pr(X = \frac{n}{2}) \to 0$ as $n \to \infty$.

Using Stirling’s approximation (specifically, $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$), we have

\[
\binom{n}{\frac{n}{2}} = \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \approx \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\sqrt{2\pi \left(\frac{n}{2}\right)} \frac{1}{2} \sqrt{2\pi \left(\frac{n}{2}\right)} \frac{1}{2}} = \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\pi n \left(\frac{n}{2}\right)^n} = \frac{\sqrt{2}}{\sqrt{\pi n} \left(\frac{1}{2}\right)^n}
\]

So,

\[
Pr(X = \frac{n}{2}) \approx \frac{\sqrt{2}}{\sqrt{\pi n} \left(\frac{1}{2}\right)^n} p^{n/2} (1 - p)^{n/2}
\]

\[
= \frac{\sqrt{2}}{\sqrt{\pi n} \left(\frac{1}{2}\right)^n} (p - p^2)^{n/2}
\]

\[
\leq \frac{\sqrt{2}}{\sqrt{\pi n} \left(\frac{1}{2}\right)^n} (1/4)^{n/2}
\]

\[
= \frac{\sqrt{2}}{\sqrt{\pi n} \left(\frac{1}{2}\right)^n} (1/2)^n
\]

\[
= \frac{\sqrt{2}}{\sqrt{\pi n}} \to 0 \text{ as } n \to \infty
\]

So we have finished the proof.
Works Cited


